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THE DUNFORD–PETTIS PROPERTY IS NOT A THREE-SPACE PROPERTY

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ABSTRACT

An example of a Banach space X is shown which does not have the Dunford–Pettis property, while some subspace Y and the corresponding quotient X/Y have the hereditary Dunford–Pettis property.

A property P is said to be a **three-space property** if, whenever a closed subspace Y of a Banach space X and the corresponding quotient X/Y have P, then X also has P. For instance, it is easy to see that reflexivity or the Schur property are three-space. A problem which has been around for some years is whether the **Dunford–Pettis property** is a three- space property (see [2] and [3] for additional information). In this paper we solve this question in the negative by showing an example of a Banach space X without the Dunford–Pettis property but such that some subspace Y and the corresponding quotient X/Y have the hereditary Dunford–Pettis property.

A Banach space X is said to have the **Dunford–Pettis property** (DPP) if any weakly compact operator $T : X \to Y$ transforms weakly compact sets of X into relatively compact sets of Y. Equivalently, given weakly null sequences (x_n)

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and (x_n^*) in X and X^* respectively, $\lim\langle x_n^*, x_n\rangle = 0$. A Banach space X is said to have the hereditary Dunford–Pettis property (DPP_h) if any closed subspace of X has the DPP. The following equivalent formulation is useful for this paper:

LEMMA: *A Banach space X has the* DPP_h *if any weakly null sequence* (x_n) *admits a sub-sequence* (x_m) *such that, for some constant* K,

$$\left\| \sum_{m=1}^{m=N} x_m \right\| \leq K.$$

The proof of which is easy using a result of Elton quoted in [2, Theorem 6].

THEOREM: *The Dunford–Pettis property is not a three-space property.*

Proof: Consider S to be Schreier's space as described in [1]. This is the space obtained by completion of the space of finite sequences with respect to the following norm:

$$\|x\|_S = \sup_{[A-\text{admissible}]} \sum_{j\in A} |x_j|,$$

where a finite sub-set of natural numbers $A = \{n_1 < \cdots < n_k\}$ is said to be admissible if $k \leq n_1$.

It can be readily verified that S is algebraically contained in c_0, that it has an unconditional basis formed by the canonical vectors (e_i), and that a sequence (x^n) of S is weakly null if and only if, for every j, the sequence $(x_j^n)_{n\in\mathbb{N}}$ is null. We show that S does not possess DPP by proving that the unit vector sequence is weakly null in S^* : this immediately follows from the estimate

$$\left\| \sum_{k=1}^{k=2^N} e_{i_k} \right\|_{S^*} \leq N.$$

We define an operator T: $l_1 \oplus_1 S \to c_0$ by means of the formula $T(y,x) = q(y) + i(x)$ where q: $l_1 \to c_0$ is a quotient map, and i denotes canonical inclusion. Obviously T is a quotient map. We only need to verify that $Ker\,T$ has DPP_h. To this end, let (y^n, x^n) be a weakly null sequence in $Ker\,T$. Since $T(y^n, x^n) = 0$ and (y^n) is norm null, one sees that also $\|x^n\|_\infty \to 0$. If $\|x^n\|_S \to 0$, then the proof ends. If not, as an application of the Bessaga-Pelczynski selection principle, we can assume that the sequence (x^n) is formed by normalized blocks of the canonical basis (e_i):

$$x^n = \sum_{i=N_n+1}^{i=N_{n+1}} \lambda_i e_i.$$

Since $\|x^n\|_S = 1$ and $\|x^n\|_\infty \to 0$, one sees that $N_{n+1} - N_n \to +\infty$. We choose a sub-sequence $(x^m) = x^{k(m)}$ as follows:

$$N_{k(m)+1} - N_{k(m)} > N_{k(m-1)},$$

and

$$\max_j |x_j^m| \leq \min_j |x_j^{m-1}|$$

where the min is taken over the non-vanishing coordinates.

We can now state that
$$\left\| \sum_{m=1}^{m=N} x^m \right\|_S \leq 2$$

because if $A = \{n_1 < \cdots < n_k\}$ with $k \leq n_1$ is admissible, and if $N_{k(m-1)} < n_1 \leq N_{k(m)}$, then

$$\sum_{j\in A} | \sum_{m=1}^{m=N} x_j^m | \leq \text{ sum of } k \text{ consecutive terms begining with } x_{n_1}^m$$
$$\leq \|x^m + x^{m+1}\| \leq 2.$$

Remark: It is worth noting that the "hereditary by quotients" Dunford–Pettis property (DPP_{hq}), in a certain sense dual of DPP_h, is a three-space property. A Banach space with this property cannot contain l_1. Since X^* has the Schur property if and only if X has DPP and does not contain l_1, a Banach space X has DPP_{hq} if and only if X^* has the Schur property, which is a three-space property.

Remark: The example $l_1 \oplus_1 S$ was used in [4] to provide a counter example to the following affirmation: the weak Banach–Saks property is a three-space property.

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